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Absolutely Continuous Conjugacies of Blaschke Products

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1. INTRODUCTION

Let $\mathbf{S}$ be the unit circle. We call a (measurable) bijection $\varphi: \mathbf{S} \to \mathbf{S}$

(finite) *nontrivial* (i.e. not 1:1 or constant) Blaschke products

$$e^{it} z^k \prod_{j=1}^{m} \left(\frac{z - a_j}{1 - \bar{a}_j z} \right), \quad |a_j| < 1,$$

which of course map $\mathbf{S}$ onto $\mathbf{S}$. Mappings $f_j: \mathbf{S} \to \mathbf{S}$, $j = 1$ and 2, are conjugate (by φ) if there is a bijection $\varphi: \mathbf{S} \to \mathbf{S}$, so that $f_2 = \varphi \circ f_1 \circ \varphi^{-1}$.

If f has no absolutely continuous conjugations to another Blaschke product, except by Mobius transformations (i.e. bilinear transformations or the complex conjugate of one) then we say that f is strongly rigid. Schub and Sullivan [17] prove

THEOREM SS. *Nontrivial Blaschke products $f: \mathbf{S} \to \mathbf{S}$, with fixed point w, $|w| < 1$, are strongly rigid.*

This was the hyperbolic case. In the case that the attractive point is on the boundary then we need not have rigidity. We find that rigidity is characterized by f being ergodic i.e. there does not exist $A \subset \mathbf{S}$, $0 < \lambda(A) < 1$, such that $f^{-1}(A) = A$. It is not hard to show that for finite Blaschke products f ergodicity is equivalent to the Julia set $J(f) = \mathbf{S}$. We prove:

THEOREM 1. *Let $f: \mathbf{S} \to \mathbf{S}$ be a nontrivial finite Blaschke product. Then f is strongly rigid if and only if the Julia set $J(f) = \mathbf{S}$.*

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Remarks. (i) The converse result cannot be regarded as new as it would follow from known results about rational dynamics. However our proof works in the transcendental case see Theorem 2, where it is new.

(ii) In later papers [8], [9], [10] we give applications to the study of Julia sets. It follows from this result that if J is a Jordan curve then it is circle line or has $\mathrm{Dim}(J) > 1$.

(iii) Although Theorem 1 does not hold for conformal self maps of the disk (the Julia set is empty), it is easy to see that a Mobius transformation $f\colon \mathbf{D} \to \mathbf{D}$ has no absolutely continuous conjugacies if and only if f is ergodic, i.e. $f^* = T \circ R \circ T^{-1}$ for some Möbius $T\colon \mathbf{D} \to \mathbf{D}$ and some rotation $R = e^{it}z$, $t/2\pi$ is irrational.

Actually we prove a weaker form of Theorem 1 for all "inner functions" f, i.e. f is bounded analytic on $\mathbf{D}$ and has radial limit of modulus 1 (a.e.) on $\mathbf{S}$. There is a well developed dynamical theory for inner functions, see [1], [2], [3], [6] and [16]. As for the rational case, a nontrivial inner function $f\colon \mathbf{S} \to \mathbf{S}$ is ergodic if there are no nontrivial invariant subset of positive measure. We say that an inner function f is rigid if there does not exist an absolutely continuous *homeomorphism* $\varphi\colon \mathbf{S} \to \mathbf{S}$ such that $f^* = \varphi \circ f \circ \varphi^{-1}$ (a.e. on $\mathbf{S}$) is an inner function.

THEOREM 2. *An inner function $f\colon \mathbf{S} \to \mathbf{S}$ is rigid if and only if it is ergodic.*

Remarks. (i) The condition that inner f is ergodic is equivalent to

$$f \text{ is "distal" on } \mathbf{D},$$

i.e. if ρ denotes the Poincare metric on the unit disk and f^n is the nth iterate of f then

$$\rho(f^n(z), f^n(w)) \to 0, \qquad \forall z \quad \text{and} \quad w \in \mathbf{D}.$$

This was proved by Aaronson [3] using a result of Pommerenke [15]

(ii) Similar results are known for Fuchsian groups G. A Fuchsian group G has nontrivial absolutely continuous conjugations if and only if G is of convergence type. One way is due to Sullivan [18], see also Tukia [20], the converse is due to Astala and Zinsmeister [4].

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2. OUTLINE OF PROOF

We now assume all our inner functions are nontrivial. First we have to show that ergodic f admitt no nontrivial conjugations. The argument now breaks into 2 cases. The simplest case is essentially done by Schub and Sullivan. If the f_j fix points in $\mathbf{D}$, conjugating each by Möbius transformations if necessary, we may assume they both fix 0. Then by Lowner's Lemma, Lebesgue measure $d\lambda$ is f-invariant, i.e. $\forall \mathbf{A} \subset \mathbf{S}$ $\lambda(f^{-1}(\mathbf{A})) = \lambda(\mathbf{A})$. An absolutely continuous bijection $\varphi: \mathbf{S} \to \mathbf{S}$, so that $f^* = \varphi \circ f \circ \varphi^{-1}$ (a.e.) is inner induces an f-invariant absolutely continuous measure $d\mu = d\lambda \circ \varphi^{-1}$ on $\mathbf{S}$. As f is ergodic by the Ergodic Theorem we have $d\mu = d\lambda$.

If f does not fix a point of $\mathbf{D}$ then there is a Denjoy-Wolff point $\zeta \in \mathbf{S}$:

$$f^n(z) \to \zeta, \qquad n \to \infty,$$

uniformly for all z in any compact subsets C of $\mathbf{D}$. Without loss of generality $\zeta = 1$. It is better to consider the equivalent problem on the upper half plane $\mathbf{H} = \{\mathrm{Re}(z) > 0\}$. Consider the Cayley Transform of the original inner function f to

$$F(z) \equiv T \circ f \circ T^{-1}, \qquad T(z) = i\,\frac{1+z}{1-z}$$

which maps $\mathbf{H}$ into itself and $F^n \to \infty$ on compact subsets. It is proved [6] that if F is ergodic then Lebesgue measure $d\lambda$ on $\mathbf{R} = \partial \mathbf{H}$ is F-invariant. However the Ergodic Theorem is not applicable in the nonfinite case. Instead we prove

THEOREM 3. *Consider an inner function $F: \mathbf{H} \to \mathbf{H}$, which is ergodic with Denjoy-Wolff point ∞. Then there is a sequence of constants $c_n \to 0$ so that*

$$\frac{1}{c_n} \int_{F^{-n}(A)} \beta(x)\, dx \to \lambda(A) \int \beta(x)\, dx$$

for any $\mathbf{A} \subset \mathbf{R}$ with compact support and $\lambda(\mathbf{A}) > 0$, and any $\beta \in L^\infty(\mathbf{R})$ with compact support.

Remarks. We are inspired by the case that

$$F(z) = z - \sum_{j=1}^{m} \frac{\mu_j}{z - x_j}, \qquad \mu_j > 0 \quad \text{and} \quad x_j \in \mathbf{R},$$

where Doering and Mane [6] prove that, if $\mu = \mu_1 + \cdots + \mu_m$,

$$\lim_{n \to \infty} \sqrt{\frac{\pi^2 \mu}{n}} \sum_{k=0}^{n} \int_{F^{-k}(A)} \beta(x)\, dx \to \lambda(A) \int \beta(x)\, dx$$

This does not hold for all the inner functions on $\mathbf{H}$ which are ergodic with Denjoy–Wolff point ∞, and Theorem 3 represents the correct generalization. One also has a counterexample of Aaronson which shows that we do not have convergence (a.e.), i.e. the Hopf Ergodic Theorem does not hold. Theorem 2 suffices to prove that for any absolutely continuous bijection $\varphi\colon \mathbf{R}\to\mathbf{R}$, such that $f^*=\varphi\circ f\circ\varphi^{-1}$ (a.e.) is inner, we have $d\lambda\circ\varphi^{-1}=c\,d\lambda$. The final step is a simple analytic continuation argument.

To prove the converse in Theorem 1 we make good use of a result of Pommerenke [15], special cases going back to Valiron:

THEOREM P. *Let $F\colon \mathbf{H}\to\mathbf{H}$ be inner and nonergodic with Denjoy–Wolff point ∞. Then there is an inner function $G\colon \mathbf{H}\to\mathbf{H}$ fixing a point of $\mathbf{H}$ and a Möbius transformation $T\colon \mathbf{H}\to\mathbf{H}$ so that*

$$G\circ F=T\circ G.$$

Furthermore we can take T to be either $z+t$ $(t\in\mathbf{R})$, or sz $(s>0)$.

In either case T generates a discrete group Γ. The rest of the construction uses the theory of quasiconformal mappings, see Lehto [11], [12]. One takes a quasiconformal mapping $\Phi\colon \mathbf{C}\to\mathbf{C}$ so that $\Phi(\mathbf{R})=\mathbf{R}$ and

(i) Φ is absolutely continous on $\mathbf{R}$,

(ii) $T=\Phi\circ T\circ\Phi^{-1}$.

Also $\Phi\circ G$ is quasiregular so there is a quasiconformal mapping $\varphi\colon \mathbf{C}\to\mathbf{C}$ with $\varphi(\mathbf{R})=\mathbf{R}$ and

$$G^*=\Phi\circ G\circ\varphi^{-1}$$

is analytic on $\mathbf{H}$. We observe that G^* is inner. Thus we put

$$\begin{aligned} F^*=\varphi\circ F\circ\varphi^{-1}&=(\varphi\circ G^{-1}\circ\Phi^{-1})\circ(\Phi\circ T\circ\Phi^{-1})\circ(\Phi\circ G\circ\Phi^{-1})\\ &=(G^*)^{-1}\circ T\circ(G^*)\end{aligned}$$

which will be the required inner function.

It remains to prove that φ is absolutely continuous and nonMöbius. This depends on suitable choice of the initial Φ as well as results from the theory of boundary values of conformal mappings.

3. NOTATION

Our sets A, B, C... are always measureable, we omit to mention it again. Complex numbers are always z, w, ζ, ω and real numbers x, y, s, t. α and β

are always real valued functions in L^{∞}, χ_A is the characteristic function of A. We do not distinguish between an analytic function f and its (radial) boundary values (even if just defined a.e.). However functions on the unit disk $\mathbf{D}$ are always $f, g, \ldots$ and $F, G\ldots$ on the upper half plane $\mathbf{H}$. Möbius transformations are denoted by $S, T\ldots$. The harmonic measure of a set $A \subset \mathbf{R}$ at a point $w \in \mathbf{H}$, with respect to $\mathbf{H}$, is $\lambda_w(A)$.

The symbols φ, Φ are restricted to bijections (not always of $\mathbf{S}$). We shall need quasiconformal mappings Φ defined on a domain Ω. Φ will have complex dilatation

$$a(z) = \frac{\bar{\partial} F}{\partial F}$$

defined a.e. (with respect to area) on Ω. Conformal mappings are also important and we make good use of the Poincare metric ρ.

4. Background

We gather necessary results from the dynamical theory of inner functions, see Aaronson [1], [2], [3], Martin [13] and especially the survey articles of Doering and Mane [6], [7]. Wherever possible we prove our results for general inner functions. Recall that Fatou proved that functions bounded and analytic on D possess radial limits

$$f(\zeta) = \lim_{r \to \infty} f(r\zeta), \qquad \zeta \in \mathbf{S} \text{ a.e..}$$

Furthermore f is inner if f is nonconstant and

$$|f(\zeta)| = 1, \qquad \zeta \in \mathbf{S} \text{ a.e..}$$

This allows us to consider the boundary map $f\colon \mathbf{S} \to \mathbf{S}$ (a.e.). Thus we say that inner functions f and f^* are conjugate (a.e.) by a bijection $\varphi\colon \mathbf{S} \to \mathbf{S}$(a.e.) if

$$f^* = \varphi \circ f \circ \varphi^{-1} \text{ (a.e. on } \mathbf{S})$$

in the sense that this is well defined and true (a.e.).

The Denjoy–Wolff Theorem asserts that if f is not Möbius then there exists a unique point $\zeta \in cL(\mathbf{D})$ so that

$$\lim_{n \to \infty} f^n(z) = \zeta$$

uniformly on all compact subsets of $\mathbf{D}$. In the case that $\zeta \in \mathbf{D}$ Lowner's Lemma (principle of harmonic measure) shows that the harmonic measure $d\lambda_\zeta$ is f-invariant.

If $\zeta \in \mathbf{S}$ then we consider the equivalent inner functions $F\colon \mathbf{H} \to \mathbf{H}$, i.e. the vertical limit $F(x) \in \mathbf{R}$, $x \in \mathbf{R}$ (a.e.). Also we assume that ∞ is the Denjoy–Wolff point. Now F has unique representation

$$F(z) = sz + t + \int_{\mathbf{R}} \frac{1 + xz}{x - z}\, d\mu\,(x)$$

for real s, t with $s \geqslant 1$ and $d\mu$ a finite (positive) measure, $d\mu \perp d\lambda$. Also $\forall \mathbf{A} \subset \mathbf{R}$,

$$\lambda(F^{-1}(A)) = \frac{1}{s}\, \lambda(A).$$

When $s > 1$, F is never ergodic and $F^n(x) \to \infty$ (a.e.). When $s = 1$ F is ergodic if and only if $\rho(F^n(z), F^n(w)) \to 0$ for any $z, w \in \mathbf{H}$. We make use of a result of [6]:

LEMMA 1. *An inner function $f = \mathbf{D} \to \mathbf{D}$ has invariant finite (positive) measure $d\mu$ on $\mathbf{S}$ which is absolutely continuous with respect to $d\lambda$ if and only if the Denjoy–Wolff point of f, ζ, say is in $\mathbf{D}$ (and in which case $d\mu = c\, d\lambda_\zeta$).*

5. Fixed Points in D

Our first result is

PROPOSITION 1. *Given an inner function $f\colon \mathbf{D} \to \mathbf{D}$ with fixed point $\zeta \in \mathbf{D}$. Suppose a bijection $\varphi\colon \mathbf{S} \to \mathbf{S}$ is absolutely continuous and there is an inner function f^* so that $f^* = \varphi \circ f \circ \varphi^{-1}$ (a.e.). Then $d\lambda_\zeta\,(\varphi) = d\lambda_\omega$ for some $\omega \in \mathbf{D}$.*

First we observe

LEMMA 2. *$d\lambda_\zeta\,(\varphi^{-1})$ is a finite f^*-invariant measure.*

For any set $\mathbf{A} \subset \mathbf{S}$,

$$\lambda_\zeta(\varphi^{-1} f^{*-1}(A)) = \lambda_\zeta(f^{-1}\varphi^{-1}(1)) = \lambda_\zeta(\varphi^{-1}(A))$$

by the f-invariance of λ_ζ.

Now if φ is absolutely continuous then so is $d\lambda_\zeta(\varphi^{-1})$. Thus by Lemmas 1 and 2 f^* is an inner function with some fixed point ω in $\mathbf{D}$. By the argument used in Lemma 2 $d\lambda_\omega \circ \varphi$ is f-invariant. Hence if φ is absolutely continuous then so is $d\lambda_\omega(\varphi)$. The theorem then follows by applying Lemma 1 again.

An extension of Proposition 1 is of some interest. We consider a bijection $\varphi: \mathbf{S} \to \mathbf{S}$ which is nonsingular, i.e. there does not exist set $\mathbf{A} \subset \mathbf{S}$:

$$\lambda(A) > 0 \quad \text{but} \quad \lambda(\varphi(A)) = 0 \quad \text{or} \quad \lambda(\varphi^{-1}(A)) = 0.$$

PROPOSITION 2. *Suppose that inner functions $f, f^*: \mathbf{D} \to \mathbf{D}$ have fixed points $\zeta, \omega \in \mathbf{D}$. Suppose also there is a bijection $\varphi: \mathbf{S} \to \mathbf{S}$ which is nonsingular and $f^* = \varphi \circ f \circ \varphi^{-1}$ (a.e.). Then $d\lambda_\zeta(\varphi) = d\lambda_\omega$.*

Conjugating f, f^* by Mobius transformations if necessary we may assume that they both fix 0. Write

$$d\lambda \circ \varphi = d\mu + dv, \qquad d\mu \ll d\lambda, \qquad dv \perp d\lambda$$

and observe that as $d\lambda \circ \varphi$ and $d\lambda$ are f-invariant so is dv and hence $d\mu$. Thus by Lemma 1 $d\mu = c\, d\lambda$, for some constant c. But then

$$d\lambda \circ \varphi \gg d\lambda,$$

and consequently $d\lambda \circ \varphi^{-1} \ll d\lambda$. But $d\lambda \circ \varphi^{-1}$ is f^*-invariant and thus by Lemma 1 $d\lambda \circ \varphi^{-1} = d\lambda$ and the proposition is proved.

6. PROOF OF THEOREM 3.

We prove a weak ergodic theorem for the inner function $F: \mathbf{H} \to \mathbf{H}$ with Denjoy–Wolff point ∞. We assume that F is ergodic and hence distal (also, see p. 10, Lebesgue measure on $\mathbf{R}$ is F-invariant)

$$\rho(F^n(z), F^n(w)) \to 0 \qquad \forall z, w \in \mathbf{H}. \tag{1}$$

Under these circumstances we prove

LEMMA 3. *For all $\alpha \in L^\infty(R)$, $\alpha \geqslant 0$ not identically zero, and $\forall z, w \in \mathbf{H}$*

$$\frac{\int_{\mathbf{R}} (\alpha \circ F^n)\, d\lambda_w}{\int_{\mathbf{R}} (\alpha \circ F^n)\, d\lambda_z} \to 1,$$

as $n \to \infty$.

Using the harmonic extension of α,

$$\int_{\mathbf{R}} (\alpha \circ F^n)\, d\lambda_w = \alpha \circ F^n(w)$$

so that

$$\frac{\int_{\mathbf{R}} (\alpha \circ F^n)\, d\lambda_w}{\int_{\mathbf{R}} (\alpha \circ F^n)\, d\lambda_z} = \frac{\alpha \circ F^n(w)}{\alpha \circ F^n(z)}.$$

Now by (1) and Harnack's inequality the righthandside tends to 1 as n tends to ∞.

Next we observe that for $\alpha \geqslant 0$, $\alpha \neq 0$ (a.e.), for all $\beta \in L^\infty(\mathbf{R})$,

$$\left| \frac{\int_{\mathbf{R}} (\alpha \circ F^n)\, \beta\, d\lambda_z}{\int_{\mathbf{R}} (\alpha \circ F^n)\, d\lambda_z} \right| \leqslant \|\beta\|_\infty . \tag{2}$$

Lemma 4. *Finite linear combinations*

$$\sum_{k=1}^{m} t_k \frac{d\lambda_{z_k}}{d\lambda_\zeta}, \ \zeta \in \mathbf{S},$$

are dense in $C(\mathbf{S})$.

Otherwise there is a finite measure $d\mu \neq 0$ on $\mathbf{S}$:

$$0 \equiv \int_{\mathbf{S}} \frac{d\lambda_w}{d\lambda_\zeta}\, d\mu\,(\zeta) \equiv u(w), \qquad w \in \mathbf{D}.$$

But u is the Poisson integral of $d\mu$ and cannot be zero!

Lemmas 4 and 3 with (2) imply that as n tends to infinity

$$\frac{\int_{\mathbf{R}} (\alpha \circ F^n) \beta\, d\lambda_z}{\int_{\mathbf{R}} (\alpha \circ F^n)\, d\lambda_z} \to \int_{\mathbf{R}} \beta\, d\lambda_z, \tag{3}$$

for continuous β with compact support, bounded positive α (not identically zero).

Now let $B \subset \mathbf{R}$ have compact support, choosing continuous $\beta > \chi_B$ we obtain from (3)

$$\overline{\lim_{n\to\infty}} \frac{\int_{\mathbf{R}} (\alpha \circ F^n)\, \chi_B\, d\lambda_z}{\int_{\mathbf{R}} (\alpha \circ F^n)\, n\lambda_z} \leqslant \lambda_z(B). \tag{4}$$

To show equality in (4) consider compact $K_m \to B$ from inside and open $V_m \to B$ from the outside. Choose continuous $\beta_m \geqslant 0$ with $\beta_m = 1$ on K_m and $\beta_m = 0$ on $\mathbf{R} - V_m$. Thus by (4)

$$\overline{\lim_{n\to\infty}} \frac{\int_{\mathbf{R}} (\alpha \circ F^n)(\chi_B - \beta_m)\, d\lambda_z}{\int_{\mathbf{R}} (\alpha \circ F^n)\, d\lambda_z} \leqslant \lambda_z(V_m - K_m).$$

Hence this and (3) gives

$$\overline{\lim_{n\to\infty}} \left| \frac{\int_{\mathbf{R}} (\alpha \circ F^n)(\chi_B)\, d\lambda_z}{\int_{\mathbf{R}} (\alpha \circ F^n)\, d\lambda_z} - \int_{\mathbf{R}} \beta_m\, d\lambda_z \right| \leqslant \lambda_z(V_m - K_m).$$

Thus in the limit as $m \to \infty$ we obtain

$$\lim_{n\to\infty} \frac{\int_{\mathbf{R}} (\alpha \circ F^n)(\chi_B)\, d\lambda_z}{\int_{\mathbf{R}} (\alpha \circ F^n)\, d\lambda_z} = \int_{\mathbf{R}} \chi_B\, d\lambda_z.$$

Now as any bounded function is the uniform limit of finite sums of $\chi_{B_|}$ the above gives

Lemma 5. *For all* $\alpha, \beta \in L^\infty(R)$, $\alpha \geqslant 0$ *not identically zero, and* $\forall z \in \mathbf{H}$

$$\frac{\int_{\mathbf{R}} (\alpha \circ F^n)\, \beta\, d\lambda_z}{\int_{\mathbf{R}} (\alpha \circ F^n)\, d\lambda_z} \to \int_{\mathbf{R}} \beta\, d\lambda_z,$$

as $n \to \infty$.

We now consider $\alpha = \chi_A$, $\beta = \chi_B\, d\lambda / d\lambda_z$ for any bounded sets A, B with $\lambda(A), \lambda(B) > 0$. Thus by Lemma 5

$$\lim_{n\to\infty} \frac{\lambda(F^{-n}(A) \cap B)}{\int_{\mathbf{R}} (\chi_A \circ F^n)\, d\lambda_z} = \lambda(B), \tag{5}$$

for any z in $\mathbf{H}$. The denominator is $\alpha \circ F^n(z)$, where α is the harmonic extension of χ_A. Fix some finite interval C say, with γ being the harmonic extension of χ_C. With this notation we prove

Lemma 6. *For any fixed* $z \in \mathbf{H}$, *as* $n \to \infty$,

$$\frac{\alpha(F^n(z))}{\gamma(F^n(z))} \to \frac{\lambda(A)}{\lambda(C)}$$

Remark. In many ways this is the observation which proves Theorem 3.

The proof is classical analysis and does not depend on an ergodic theorem although it looks like it must.

In fact all we use is $z_n = F^n(z) \to \infty$ in $\mathbf{H}$. Now we take a finite interval $A = [x, y]$. Then

$$\begin{aligned}\alpha(z) &= \frac{1}{\pi} \arg \left\{ \frac{z-x}{z-y} \right\} \\ &= \frac{y-x}{\pi} \operatorname{Im} \left\{ \sum_{j=1}^{\infty} \frac{x^{j-1} + \cdots + y^{j-1}}{jz^j} \right\}, \qquad \text{for} \quad |z| > \max |x|, |y|.\end{aligned}$$

Then if $z_n = r_n \exp(it_n)$, $0 < t_n < \pi$, $r_n \to \infty$ we find that

$$\alpha(z) = \frac{y-x}{\pi} \frac{\sin(t_n)}{r_n} \{1 + \varepsilon_n\},$$

where

$$\begin{aligned}\varepsilon_n &< \sum_{j=2}^{\infty} \frac{|x|^{j-1} + \cdots + |y|^{j-1}}{jr_n^{j-1}} \left| \frac{\sin(jt_n)}{\sin(t_n)} \right| \\ &< \sum_{j=1}^{\infty} \frac{|x|^{j-1} + \cdots + |y|^{j-1}}{r_n^{j-1}} \\ &= \frac{\delta r_n}{(r_n - \delta)^2}, \qquad \delta = \max\{|x|, |y|\}.\end{aligned}$$

Assuming that we shall only consider sets in some fixed interval $(-\delta, \delta)$ shows that for an otherwise arbitrary set A

$$\alpha(z) = \frac{\lambda(A)}{\pi} \frac{\sin(t_n)}{r_n} \{1 + \varepsilon_n\},$$

which with a similar result for $\gamma(z)$ proves the lemma.

Lemma 6 with (5) shows that

$$\begin{aligned}\lim_{n \to \infty} \frac{\lambda(F^{-n}(A) \cap B)}{\int_{\mathbf{R}} (\chi_C \circ F^n) \, d\lambda_z} \lambda(C) &= \lim_{n \to \infty} \frac{\lambda(F^{-n}(A) \cap B)}{\int_{\mathbf{R}} (\chi_A \circ F^n) \, d\lambda_z} \lambda(A) \\ &= \lambda(A) \, \lambda(B).\end{aligned}$$

Thus we have proved Theorem 3 for $\beta = \chi_B$ with

$$c_n = \frac{\int_{\mathbf{R}} (\chi_C \circ F^n) \, d\lambda_z}{\lambda(C)}.$$

The rest of the proof follows the earlier approximating arguments.

7. Ergodic Case II

We now make use of Theorem 3 to prove:

Theorem 4. *Suppose that* $F, F^*: \mathbf{H} \to \mathbf{H}$ *are ergodic and inner with Denjoy–Wolff point* ∞. *Let* $F^* = \varphi \circ F \circ \varphi^{-1}$ *(a.e.) for some absolutely continuous bijection* $\varphi: \mathbf{R} \to \mathbf{R}$. *Then*

$$d\lambda \circ \varphi = c\, d\lambda$$

for some constant c.

We may restrict ourselves to bounded sets B which φ and φ^{-1} map to bounded sets and $d\mu = d\lambda \circ \varphi|_B = \beta\, d\lambda$ for β bounded on B. Consider a fixed bounded set A of the same type, and assume A and B have positive measure. Let $A^* = \varphi(A)$, $B^* = \varphi(B)$. By Theorem 3,

$$\frac{1}{c_n} \int_{F^{-n}(A)} \beta(x)\, dx \to \lambda(A)\, \mu(B)$$

But as F^* is ergodic, applying Theorem 3 again, there is a sequence $c_n^* \to 0$, so that

$$\frac{1}{c_n^*} \int_{F^{*-n}(A^*)} \chi_{B^*}\, dx \to \lambda(A^*)\, \lambda(B^*) = \mu(A)\, \mu(B).$$

Now the change of variable formula for measures shows

$$\int_{F^{*-n}(A^*)} \chi_{B^*}\, dx = \int_{F^{-n}(A)} \chi_B \beta\, dx$$

and thus

$$\lim_{n \to \infty} \frac{c_n}{c_n^*} = c = \frac{\mu(A)}{\lambda(A)},$$

for all such A. However as μ is absolutely continuous we obtain

$$\lambda \circ \varphi(A) = c\lambda(A),$$

for all bounded subsets $A \subset \mathbf{R}$ with positive measure.

8. Proof of Theorem 1 (part I)

We now deal with the case of finite Blaschke products (or rational inner functions). In the ergodic case we have reduced the problem to the following cases:

(i) $f, f^*: \mathbf{D} \to \mathbf{D}$, fix 0, φ preserves Lebesgue measure on $\mathbf{S}$,

(ii) $F, F^*: \mathbf{H} \to \mathbf{H}$, fix ∞, φ preserves Lebesgue measure on $\mathbf{R}$.

Schub and Sullivan uses estimates on the second derivative together with the fact that f, f^* are expanding to show that φ is an isometry. In case (ii) we use an argument inspired by another result of [17], using analytic continuation to show the same. Actually our method also works in case (i) but we do not show the details.

The real analytic functions

$$u = \left|\frac{dF}{dx}\right|, \qquad u^* = \left|\frac{dF^*}{dx}\right|$$

satisfy

$$u^* = u \circ \varphi^{-1} \text{ (a.e.)},$$

as φ is absolutely continuous with derivative 1 (a.e.). Now as F has poles on the real line u is not constant, hence the function

$$u^{-1} \circ u^* = h$$

is defined and locally analytic except at a countable number of critical points. Thus there are sets E of positive measure so that *on* E φ^{-1} is equal to a branch of h. But then by (ii) $|h'| = 1$ (a.e.) on E. Thus on E, at least,

$$\varphi(x) = \pm x + t$$

for some sign and t depending on E. Let us consider the class L of h arising in this way. By analytic continuation, for any h in L,

$$u^* \circ h(x) = u(x), \qquad \text{for all } x.$$

We now prove

LEMMA 7. *With the above notation, L contains a single function.*

Suppose $x + t, x + s \in L$ for $s \neq t$. So

$$u^*(x+s) = u(x) \;\&\; u^*(x+t) = u(x) \qquad \text{for all } x.$$

Then u has nonzero period $s - t$ which is impossible as F is rational. If $-x + t$, $-x + s \in L$ for $s \neq t$, then

$$u^*(-x+s) = u(x) \;\&\; u^*(-x+t) = u(x) \qquad \text{for all } x.$$

Thus

$$u^*(x+s-t)=u(x)\ \&\ u^*(x+t-s)=u(x) \qquad \text{for all } x.$$

So $2(t-s)$ is a period of u, also impossible. If $x+t$, $-x+s \in L$ then $-x+s-t \in L$. As before this is impossible provided t is nonzero. If $t=0$ then $u=u^*$ and so $u(-x+s)=u(x)$. Thus in all cases the only possibilities is that either L contains exactly one function h or else x and $-x+s \in L$. We show that we cannot have x and $-x+s \in L$ if F is ergodic. Consider the set A where $\varphi = x$ with complement B where $\varphi = -x+s$. We show that A and B are F invariant (a.e). Otherwise there is a set of positive measure where $F^*(x)=F(-x+s)$, and hence $F^*(x)=F(-x+s)$ for all x (by analytic continuation again). But both F and F^* are asymptotic to x at ∞ and we cannot have one like $-x$. Now F is ergodic and F invariant sets have measure zero or the complement has measure zero. Thus A or B has measure zero. Therefore L contains exactly one function h which is therefore equal to φ. In particular φ is an isometry.

We end this section by sketching the proof for case (i). As above φ is equal to isometries T_A on sets A of positive measure which partition the circle. We use the fact that f is a rotation or strictly expanding, i.e. $\exists c>1: \lambda(f(A))>c\lambda(A)$ for all sets A. Thus by taking enough iterates any A eventually covers most of the circle, intersecting most of the other A's. Thus $f^{*n}=T_{A'}f^nT_{A^{-1}}$ if $f^n(A)$ intersects A' on a set of positive measure. In particular $f^n(A)$ intersects A on a set of positive measure so we have $f^{*n}=T_Af^nT_{A^{-1}}$ (analytic continuation again). Equating the two expressions for f^{*n} shows that all the isometrics are the same and φ is equal everywhere to one isometry.

It remains to observe that ergodicity is equivalent to $J(f)=\mathbf{S}$. Certainly if $J(f)$ is a proper subset of the circle then f cannot have its Denjoy–Wolff point $\zeta \in \mathbf{D}$. So $\zeta \in \mathbf{S}$ is the attractive point on the boundary of a parabolic domain $\mathbf{C}-J(f)$ or the attractive point inside a domain $\mathbf{C}-J(f)$. In the first case the classification theory of Parabolic domains shows that ζ is the endpoint of one of the complementary arcs α of $J(f)$. Furthermore we can choose a small closed subarc $\beta \subset \alpha$ with $f^n(\beta)$ all subarcs of α tending uniformly to ζ, $n=1, 2, \ldots$. For small enough β the set $A=\bigcup_{-\infty}^{+\infty} f^n(\beta)$ is a proper closed subset of the circle with $\lambda(A)<1$. Also $f^{-1}(A)=A$. Thus f is not ergodic. In the other case, that ζ is an attractive point of $\mathbf{C}-J(f)$ an almost identical argument also shows that f is not ergodic. Conversely, if $J(f)=\mathbf{S}$ and f has its Denjoy–Wolff point inside $\mathbf{D}$ then f is ergodic (by Aaronson). If $z \in \mathbf{S}$ then $\mathbf{D}$ is a parabolic domain. A result of Doering and Mane, see p. 24 [6], shows that f is ergodic.

This completes the first part of the proof of Theorem 1 (also 2).

9. Quasiconformal Deformations

For the convenience of the reader we give some backgound on quasiconformal mappings and inner functions. We now restrict ourselves to non-ergodic inner functions $F\colon \mathbf{H} \to \mathbf{H}$ and show they have nontrivial absolutely continuous conjugacies.

A homeomorphism Φ of a domain Ω is quasiconformal if it has generalized derivatives in L^2_{loc} which satisfy the Beltrami equation

$$\frac{\bar{\partial}\Phi}{\partial\Phi} = a(z), \qquad \text{(a.e.) on } \Omega, \tag{6}$$

where the complex dilatation $a(z)$ satisfies $\|a\|_\infty < 1$. Conversely given any such $a(z)$ there a homeomorphism Φ of Ω with generalized derivatives in L^2_{loc} which satisfy (6).

Theorem P shows that

$$F = G^{-1} \circ T \circ G \tag{7}$$

for some inner function $G\colon \mathbf{H} \to \mathbf{H}$ (G fixes a point of $\mathbf{H}$) and Möbius transformation $T\colon \mathbf{H} \to \mathbf{H}$ which has no fixed points in $\mathbf{H}$ and generates a discrete group Γ. As we are dealing with nonconstant analytic functions the right hand side of (7) is understood in a local sense. The result of the previous theory is that there is analytic continuation to a global function F. Of course for arbitrary G equation (7) does not give an analytic function F. Equation (7) shows that in order for F to be analytic the singularities of G must display certain types of invariances in T. In particular G is never rational (except in trivial cases). For example if $T = z+1$ we might define G by its Riemann surface R. We require R to be a covering of the upper half plane having algebraic singularities of order 2 at $w = i, i+1, i+2, \ldots$, and other algebraic singularities of order 4 at $w = i-1, i-2, \ldots$. The resulting function F has exactly one critical point of order $4/2 = 2$ at $z = G^{-1}(1-i)$.

We now consider the construction of certain Γ-equivariant quasiconformal mappings $\Phi\colon \mathbf{H} \to \mathbf{H}$, i.e. so that

$$S = \Phi \circ T \circ \Phi^{-1}$$

is a Möbius transformation of $\mathbf{H} \to \mathbf{H}$. This is well understood in the theory of Teichmuller spaces, see [11]. A necessary and sufficient condition is that the complex dilatation $a(z)$ of Φ satisfy

$$a \circ T(z) \frac{\overline{T'(z)}}{T'(z)} = a(z) \tag{8}$$

which ensures T-equivariance and

$$a(\bar{z}) = \overline{a(z)} \tag{9}$$

which ensures that Φ may be chosen to map $\mathbf{R}$ onto itself. Finally without loss of generality, by composing S by a Moblus transformation if necessary, we may assume $S = T$. Our $a(z)$ is constructed by first choosing a small open set Δ in a fundamental region of $\Gamma = \langle T \rangle$ and taking any function $a(z)$ supported on Δ with $\|a\|_\infty \ll 1$. Then a is extended to the set $U = \{T^n(\Delta) : T \in \Gamma\}$ and its complex conjugate by means of (8) and (9). At all other points of the plane a is set equal to zero.

While the mapping $\Phi \circ G$ is usually not analytic, it is quasiregular on $\mathbf{H}$ with complex dilatation

$$b(z) \equiv a \circ G(z) \frac{\overline{G'(z)}}{G'(z)}, \qquad \mathrm{Im}(z) > 0, \tag{10}$$

which is seen by the composition formula, [12] p. 183. Let $\varphi : \mathbf{C} \to \mathbf{C}$ be the quasiconformal mapping with complex dilatation

$$c(z) = \begin{cases} b(z), & \mathrm{Im}(z) > 0 \\ \overline{b(\bar{z})}, & \mathrm{Im}(z) < 0 \end{cases}.$$

The symmetry in the definition of c allows φ to be chosen to map $\mathbf{H}$ onto $\mathbf{H}$, and fix 0, 1, ∞.

Consider the functions

$$G^*(z) \equiv \Phi \circ G \circ \varphi^{-1}$$

and

$$F^*(z) \equiv \varphi \circ F \circ \varphi^{-1}.$$

Lemma 8. *G^*, F^* are analytic on the upper half plane.*

For G^* the composition formula for dilatations shows that complex dilatation is zero. For F^* we observe

$$\begin{aligned} F^*(z) &\equiv \varphi \circ G^{-1} \circ T \circ G \circ \varphi^{-1} \equiv \varphi \circ G^{-1} \circ \Phi^{-1} \circ \Phi \circ T \circ \Phi^{-1} \circ \Phi \circ G \circ \varphi^{-1} \\ &\equiv G^{*-1} \circ T \circ G^* \end{aligned}$$

is analytic.

Next we show that G^*, F^* can be chosen to be “inner” on the upper half plane. If F is rational this is of course an immediate consequence of the fact that its Riemann surface is an n-sheeted covering.

LEMMA 9. *There exists μ: F^*, G^* are "inner" on the upper half plane.*

It is easier to deal with the equivalent result on the unit disk. Thus f is an inner function of the disk and $\varphi, \Phi: \mathbf{D} \to \mathbf{D}$ quasiconformal mappings so that

$$g^*(z) \equiv \Phi \circ g \circ \varphi^{-1}$$

is analytic on the unit disk. First we give the result for f a finite Blaschke product, i.e., rational. Now from the proof of Theorem P we have g^* is inner if and only if f^* is inner. However any quasiconformal deformation of a rational function is rational.

In general we have the problem that a qc deformation of some inner functions need not give an inner function. (However if f has radial limits w, $|w|=1$ except for a set E of dimension $Dim(E)<1$ or if just the set R of radial limits $w \in \mathbf{D}$ satisfy $cap(R)=0$ then f^* will be inner). However, in our problem it merely suffices to find some μ so that Φ preserves harmonic measure 0.

Actually it suffices that g has some nontrivial qc deformation Φ_1 to an inner function. First we restrict the dilatation to a small disk Δ and extend in an invariant way to obtain an invariant deformation Φ. We now see that this can be done to preserve harmonic measure and thus f^*, g^* will be inner. Now inside and outside Δ and its images, the two deformations are equivalent up to a conformal mapping and hence preserve harmonic measure. We have to worry about Φ transforming a subset of $\partial\Delta$ to a set of positive harmonic measure (or vice versa). An argument on almost all circles shows we can choose a Δ where this cannot happen.

Hence any inner function which has a nontrivial deformation to an inner function may be chosen to have a deformation of the required type to some inner function. Thus f^*, g^* may be chosen to be inner.

10. GOOD BOUNDARY POINTS FOR INNER FUNCTIONS

Recall that φ has dilatation $c(z)$ supported on the set

$$C = G^{-1}\left\{\bigcup_{T \in \Gamma} T(\Delta)\right\}.$$

We are interested in the set V of real numbers x so that there is a sequence z_n in C with

$$z_n \to x, \text{ in some Stolz Angle of } \mathbf{H}.$$

Now the set

$$\bigcup_{T\in\Gamma} T(\Delta)$$

at most has two cluster points L on $\partial\mathbf{H}$ (including ∞). By Lowner's Lemma

$$\lambda(G^{-1}(L))=0.$$

Thus by recalling that an inner function G has real valued boundary values (a.e.) we have

LEMMA 10. *With the above notation* $\lambda(V)=0$.

Conversely as $\Phi^{-1}\circ G^*\circ\varphi=G$ we see that the dilatation of φ^{-1} is supported on a set which has limit points through Stolz Angles on a set $V^*\subset\mathbf{R}$ which also satisfies $\lambda(V^*)=0$. Furthermore

$$V^*=\varphi(V),$$

so φ maps V to a set of zero measure.

11. CONFORMAL MAPPING

To complete the proof that φ is absolutely continuous we make of two results on the harmonic measure of sets E on the boundary of a simply connected domain. The first result is due to Riesz, see [14]:

LEMMA 11. *Let Ω be a simply connected domain bounded by a rectifiable Jordan curve. Then harmonic measure and length are mutually absolutely continuous.*

The second result is essentially due to Lowner (again), see [14]:

LEMMA 12. *Let $\Omega\subset\mathbf{H}$ be a simply connected domain bounded by a Jordan curve. Then if a set $E\subset\partial\Omega\cap\mathbf{R}$ has positive harmonic measure then $\lambda(E)>0$.*

Finally we finish the proof that φ is absolutely continuous on $\mathbf{R}$. Suppose that K is any subset of R which has positive measure. We show that K has a subset E of positive measure such that $\lambda(\varphi(E))>0$. A similar result for φ^{-1} indeed proves absolute continuity of φ.

For every point x of $K-V$, where V comes from Lemma 11, there is a cone

$$U(x)=\{z\in\mathbf{H}\colon \theta<\arg(z-x)<\pi-\theta,\ |z-x|<r(x)\}$$

in H with vertex x in which φ is conformal, for some positive number $\theta = \theta(x) < \pi/2$. The angle θ of course depends on x. However by the standard measure theoretic arguments $K - V$ has a subset L of positive measure such that there is a fixed positive number $\tau < \pi/2$ so that for all x in L $\theta(x) < \tau$. Consider the open set

$$\bigcup_{x \in L} U(x)$$

which has at least one component Ω which meets $\mathbf{R}$ in a set E of positive measure. Following the classic Privalov construction we may take $\partial\Omega$ to be rectifiable (see Fig. 1).

Now by Riesz', Lemma 11, E has positive harmonic measure (at any point of Ω with respect to Ω). Also φ is conformal on Ω and the image of E is a set $E^* \subset \mathbf{R}$, thus as $\Omega^* = \varphi(\Omega) \subset \mathbf{H}$, Lemma 12 implies that $\lambda(E^*) > 0$. This completes the proof that φ is absolutely continuous.

To complete the proof of Theorem 1 we have only to show that φ is not a Mobius transformation. Here we use the rich supply of deformations.

12. φ Is Nontrivial

It is very easy to see that φ is not a Mobius Transformation. It was constructed to fix 0, 1, ∞. So if it is Mobius then $\varphi = x$ on $\mathbf{R}$. On the open set

$$A = \mathbf{C} - G^{-1}\left\{\bigcup_{T \in \Gamma} T(\Delta)\right\}$$

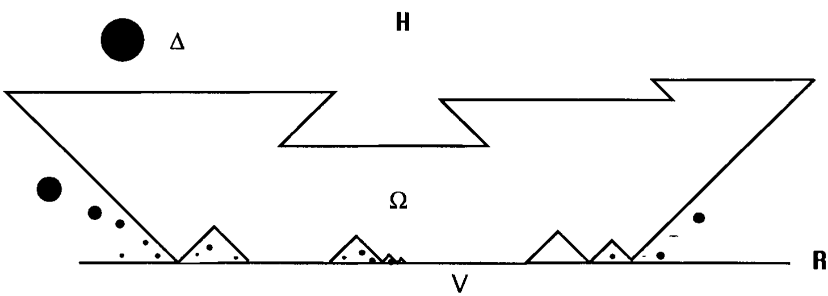

FIGURE 1

φ is nonformal. The problem is that

$$G^{-1}\left\{\bigcup_{T\in\Gamma} T(\Delta)\right\}$$

can cluster on a set dense in $\mathbf{R}$. Now A is connected and by similar considerations to Lemmas 11 and 12 $\mathbf{R}-V$ has positive harmonic measure with respect to A. Thus if $\varphi=z$ on a boundary set of positive harmonic measure we see that $\varphi\equiv z$ on A. We must now arrange the initial complex dilatation $a(z)$ so that this is impossible.

Fix some component D of $G^{-1}(\Delta)$. Then the dilatation $c(z)$ may be freely chosen on D. The formula in the previous section then generates the dilatation $c(z)$ on all of $G^{-1}(\Delta)$. We choose the dilatation $c(z)$ on D so that the quasiconformal solution ψ of the Beltrami equation

$$\frac{\bar{\partial}\psi}{\partial\psi}=\begin{cases} c(z), & z\in D\\ 0, & \text{otherwise}\end{cases}$$

is not analytic on ∂D, more precisely $\psi(\partial D)$ is not an analytic subarc. As the other components of $G^{-1}(\Delta)$ are disjoint from D standard quasiconformal theory shows that $\varphi=\varphi\circ\psi$ where φ is conformal on $\psi(\partial D)$ at least. Thus $\varphi(\partial D)$ is not an analytic arc either. Hence ψ cannot be Mobius on the domain A.

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